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PAPER

BA’ZI KLASSIK INTEGRALLARNI HISOBLASH

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Abstract

Ushbu maqolada ba’zi klassik integrallarni hisoblash usullari koʻrib chiqilgan. Misollar orqali ularni yechishning samarali metodlari —almashtirish, qismlarga ajratish va trigonometrik usullar asosida tahlil qilingan. Natijalar integral hisoblash koʻnikmalarini mustahkamlashga xizmat qiladi.

Key words: parametr, integral, Dirixle integrali, Laplas integrali, Eyler-Presson integrali, Frenel integrallari, Frullani integrali.

Kirish

Tatbiqlari amaliy masalalarda koʻp uchraydigan, mashhur matematikaning nomlafri bilan ataladigan bir nechta integrallarni hisoblash bilan shugʻullanamiz.

1. Dirixle integrali (Peter Grustov dejen-Dirixle (1805-1859)-nemis matematigi)

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

Integralni hisoblash uchun $\int_0^\infty e^{-xy}\sin x\,dx$ yordamchi integralni qaraymiz. Uni ikki marta bo’laklab integrallab, $\int_0^\infty e^{-xy}\sin x\,dx$ ga nisbatan tenglama hosil qilib,

$$\int_0^\infty e^{-xy}\sin x\,dx = \frac{y}{1+y^2}$$

ni topamiz.

$\delta > 0$ da istalgan $[\delta, N]$ da y parametr bo’yicha (1) integral tekis yaqinlashadi. Bu tekis yaqinlashish uchun Veyershrass alomatidan kelib chiqadi, chunki

$$|e^{-xy}\sin x| < e^{-\delta x}, \quad \int_0^\infty e^{-\delta x}\,dx = \frac{1}{\delta}$$

Parametr bo’yicha integrallash formulasi (1) integralga qo’llab, $\int_\delta^N \frac{dy}{1+y^2} = \arctan N - \arctan\delta$

$= \int_0^\infty dx \int_\delta^N e^{-xy}\sin x\,dy$
$= \int_0^\infty \frac{e^{-\delta x} - e^{-Nx}}{x}\sin x\,dx$
ni hosil qilamiz. $|\sin x| \le x$ bo’lg’ani uchun

$$\left|\int_0^\infty \frac{e^{-Nx}\sin x}{x}\,dx\right| \le \int_0^\infty e^{-Nx}\,dx = \frac{1}{N} \text{ bo’ladi.}$$

$N \to \infty$ da yuqoridagi tenglikda limitga o’tib,

$$\frac{\pi}{2} - \arctan\delta = \int_0^\infty e^{-\delta x}\sin x\,dx = \int_0^\infty \frac{\sin x}{x}\,dx$$

tenglikdan va $\delta \to 0$ da limitga o’tib,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

tenglikni hosil qilamiz. Bu Dirichlet integrallash qiymati bo’ladi.

2. Laplas integrallari (Pyer Simon Laplas (1749-1827) –mashhur fransuz astronomi, fizigi va matematigi)

$$I_1(y) = \int_0^\infty \frac{\cos xy}{1+x^2}\,dx, \quad I_2(y) = \int_0^\infty \frac{x\sin xy}{1+x^2}\,dx$$

integrallar Laplas integrallari deb ataladi. Ularni hioblaymiz:

$y \ge \delta > 0$ bo’lsin. U holda ikkalasi integral ham $y \in [\delta, +\infty)$ parametr bo’yicha Dirixle alomatiga ko’ra tekis yaqin-

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lashadi. Chunki, $\cos xy$, $\sin xy$ lar chegralangan boshlang'ich funksiyalarga ega $\frac{1}{1+x^2}$ va $\frac{x}{1+x^2}$ lar $x \to +\infty$ da nolga intilishadi.

Bundan tashqari, $x \geq 1$ da

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = -\frac{2x}{(1+x^2)^2} \leq 0, \quad \frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{1-x^2}{(1+x^2)^2} \leq 0.$$

$I_1(y)$ ni parametr bo'yicha differensiallab,

$$\frac{dI_1(y)}{dy} = -\int_0^\infty \frac{x\sin xy}{1+x^2}dx = -I_2(y), \quad \delta \leq y < \infty 2$$

ga ega bo'lamiz. $y \in [\delta, +\infty)$ da $I_2(y)$ tekis yaqinlashgani uchun parametr bo'yicha differensiallash qonuni hisoblanadi. $I_2(y)$ ning bosilasini topish uchun $y \geq \delta > 0$ da

$I_2(y) = \int_0^\infty \frac{x\sin xy}{1+x^2}dx$

$= \int_0^\infty \left(\frac{1}{x} - \frac{1}{x(1+x^2)}\right)\sin xy\, dx$

$= \int_0^\infty \frac{\sin xy}{x}dx - \int_0^\infty \frac{\sin xy}{x(1+x^2)}dx$

$= \int_0^\infty \frac{\sin t}{t}dt - \int_0^\infty \frac{\sin xy}{x(1+x^2)}dx$

$= \frac{\pi}{2} - \int_0^\infty \frac{\sin xy}{x(1+x^2)}dx 3$

ga ega bo'lamiz. Parametrga bog'liq xosmas integrallarini differensiallash qoidasiga binoan

$$\frac{dI_2(y)}{dy} = -\int_0^\infty \frac{\cos xy}{1+x^2}dx = -I_1(y), \quad y \in [\delta, +\infty)4$$

$I_1'(y) = -I_2, \quad I_2'(y) = -I_1(y), \quad I_1''(y) - I_1(y) = 0$

Oxirgi differensial tenglamani yechib,

$I_1(y) = C_1e^{-y} + C_2e^{y}, \quad y \in [\delta, +\infty), \quad C_1, C_2$ – ixtiyoriy o'zgarmaslar, (5) ni hisil qilamiz

$$|I_1(y)| = \left|\int_0^\infty \frac{\cos xy}{1+x^2}dx\right| \leq \int_0^\infty \frac{|\cos xy|}{1+x^2}dx \leq \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

munosabatdan $[\delta, \infty)$ da $I_1(y)$ chegaralangan funksiya ekanligini, e^y – chegaralanmaganligini ko'rib, (5) da $C_2 = 0$ ekanligini aniqlaymiz.

Shunday qilib,

$$I_1(y) = C_1e^{-y}, \quad I_2(y) = -I_1'(y) = C_1e^{-y}, \quad y \in [\delta, \infty)6$$

larni bosil qilamiz.

δ – ixtiyoriy musbat son bo'lgani uchun $y > 0$ da

$$I_1(y) = C_1e^{-|y|}, \quad I_2(y) = C_1 \mathrm{sign} y e^{-|y|}, \quad y \neq 0 8$$

ko'rinishda yozib olamiz.

Ixtiyoriy C_1 o'zgarmasni aniqlash uchun $I_1(y)$ Laplas integrallari $(-\infty, +\infty)$ da y parametr bo'yicha tekis yaqinlashishidan foydalanamiz. Shuning uchun $I_1(y)$ $y = 0$ nuqtada uzluksiz funksiya. Demak

$$\frac{\pi}{2} = \int_0^\infty \frac{dx}{1+x^2} = I_1(0) = \lim_{y\to 0} I_1(y) = \lim_{y\to 0} C_1e^{-y} = C_1$$

Endi (8) formulani istalgan $y \in R$ da

$$\int_0^\infty \frac{\cos xy}{1+x^2}dx = \frac{\pi}{2}e^{-|y|}, \quad \int_0^\infty \frac{x\sin xy}{1+x^2}dx = \frac{\pi}{2}\mathrm{sign} y e^{-|y|} 9$$

larni beradi. $y = 0$ da ham (9) ni to'g'ri riligini bevosita tekshiriladi.

3. Eyler-Puasson integrallari (Leonard Eyler (1707-1783)-mashhur rus olimi, asli Shveytsariyalik, Simeon Deni Puasson(1781-1840)- fransuz matematigi, mexanigi, fizigi)

Bu integral

$$I = \int_0^\infty e^{-t^2}dt$$

ko'rinishga ega bo'lib, bu integral ehtimolliklar nazariyasida ko'p uchraydi.

I integralda $t = xy, y > 0$ almashtirish bajaramiz. U holda

$$I = y\int_0^\infty e^{-x^2y^2}dx$$

hosil bo'ladi. Bu tenglikni e^{-y^2} ga ko'paytirib, uni bo'yicha 0 dan $+\infty$ gacha integrallasak.

$$I^2 = \int_0^\infty te^{-y^2}dy = \int_0^\infty dy\int_0^\infty ye^{-y^2(1+x^2)}dx 10$$

ga ega bo'lamiz.

Bu integraldi integrallash tartibini almashtirib.

$$I^2 = \int_0^\infty dx\int_0^\infty ye^{-y^2(1+x^2)}dy = -\int_0^\infty \frac{e^{-y^2(1+x^2)}}{2(1+x^2)}\Bigg|_0^\infty dx = \frac{1}{2}\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{4},$$

ni bundan esa

$$I = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

ni hosil qilamiz.

4. Frenel integrallari. Optikada keng tatbiqini topgan va Frenel integrallari deb atalaychi

$$I_1 = \int_0^\infty \sin x^2 dx, \quad I_2 = \int_0^\infty \cos x^2 dx$$

integrallarni hisoblaymiz. $x^2 = t$ almashtirish yordamida ular

$$I_1 = \frac{1}{2}\int_0^\infty \frac{\sin t}{\sqrt{t}}dt, \quad I_2 = \frac{1}{2}\int_0^\infty \frac{\cos t}{\sqrt{t}}dt$$

ko'rinishdagiga keltiriladi. Bu integrallar Dirixle alomatiga ko'ra yaqinlashuvi bo'ladi. I_1 da integral ostidagi funksiya $x = 0$ da nolga teng deb hisoblaymiz. I_2 ni hisoblaymiz. Agar $t > 0$ bo'lsa,

$$\int_0^\infty e^{-tx^2}dx = \frac{1}{\sqrt{t}}\int_0^\infty e^{-(\sqrt{t}x)^2}d(\sqrt{t}x) = \frac{\sqrt{\pi}}{2\sqrt{t}}$$

tenglik Eyler-Puasson integraliga ko'ra o'rnatilgan bo'lib, bundan

$$\frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-tx^2}dx, \quad I_1 = \frac{1}{\sqrt{\pi}}\int_0^\infty \sin t\, dt\int_0^\infty e^{-tx^2}dx$$

ni hosil qilamiz.

$\beta \in R^+$ parametrga bog'liq quyidagi

$$\phi(\beta) = \frac{1}{2}\int_0^\infty e^{-\beta t}\frac{\sin t}{\sqrt{t}}dt = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\beta t}\sin t\, dt\int_0^\infty e^{-tx^2}dx$$

integralni qaraymiz.

$\left|e^{-\beta t}e^{-x^2}\sin t\right| \leq e^{-\beta t}, \quad \forall t \in R^+,$ tengsizlikdan va $\forall\beta > 0$ da $\int_0^\infty e^{-\beta t}dt = \frac{1}{\beta}$

integralning yaqinlashuvchiligidan

$$\int_0^\infty e^{-\beta t}e^{-tx^2}|\sin t|dt, \quad \int_0^\infty e^{-\beta t}e^{-tx^2}|\sin t|dx$$

integrallar mos ravishda x va t o'zgaruvchilar bo'yicha tekis yaqinlashuvchi bo'lib, integrallash tartibini almashtirishimiz mumkin:

$$\phi(\beta) = \frac{1}{\sqrt{\pi}} \int_0^\infty dx \int_0^\infty e^{-(\beta+x^2)t} \sin t dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(\beta+x^2)t}(\cos t + (\beta + x^2)\sin t)}{1 + (\beta + x^2)^2}\Bigg|_{t=0}^{t=\infty} dx$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dx}{1 + (\beta + x^2)^2}$$

$$\left|e^{-\beta t} \frac{\sin t}{\sqrt{t}}\right| \leq \frac{\sin t}{\sqrt{t}}, \quad \forall\beta \geq 0 \text{ va } \forall t > 0,$$

tengsizlikdan va $\int_0^\infty \frac{\sin t}{\sqrt{t}} dt$ integrali

yaqinlashuvchiligidan $\phi(\beta)$ integral $\beta \geq 0$ da tekis yaqinlashuvchiligi va bundan esa $\forall\beta > 0$ da uzluksiz ekanligi kelib chiqib, $\phi(\beta)$ integralda $\beta \to +0$ bo'lganda limitga o'tish mumkin:

$$I_1 = \phi(+0) = \lim_{\beta\to 0} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dx}{1 + (\beta + x^2)^2}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dx}{1 + x^4}$$

$$= 1 \overline{\sqrt{\pi}\left(\frac{1}{\sqrt{2}} \ln \frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\left(\arctan(x\sqrt{2}+1)+\arctan(x\sqrt{2}-1)\right)\right)\Big|_0^\infty}$$

$$= \frac{1}{2\sqrt{2}}\pi = \frac{1}{2}\sqrt{\frac{\pi}{2}}, \text{ ya'ni } I_1 = \frac{1}{2}\sqrt{\frac{\pi}{2}} \text{ bo'ladi.}$$

Shunga o'xshash mulohaza qilib, va

$$\frac{1}{\sqrt{\pi}} \int_0^\infty dx \int_0^\infty e^{-(\beta+x^2)t} \cos t\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(\beta+x^2)t}(\sin t - (\beta + x^2)\cos t)}{1 + (\beta + x^2)^2}\Bigg|_{t=0}^{t=\infty} dx$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\beta + x^2}{1 + (\beta + x^2)^2} dx$$

tenglikni e'tiborga olib,

$$I_2 = \frac{1}{\sqrt{\pi}} \int_0^\infty \lim_{\beta\to 0} \frac{\beta + x^2}{1 + (\beta + x^2)^2} dx$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{x^2\, dx}{1 + x^4}$$

$$= \frac{1}{2\sqrt{\pi}}\left(\frac{1}{2\sqrt{2}} \arctan \frac{x^2 - 1}{x\sqrt{2}} + \frac{\pi}{4\sqrt{2}} \text{signx}\right.$$

$$\left.+ \frac{1}{4\sqrt{2}} \ln \frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1}\right)\Bigg|_0^\infty = \frac{1}{2\sqrt{2}}\pi$$

ni hosil qilamiz. Shunday qilib,

$$I_1 = I_2 = \frac{1}{2}\sqrt{\frac{\pi}{2}} \text{ ekan.}$$

Frullani integrallari: Faraz qilaylik $[0;+\infty)$ da aniqlangan $f(x)$ funksiya uzluksiz va $\forall A > 0$ da $\int_0^\infty \frac{f(x)}{x} dx$ yaqinlashsa, u holda

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = f(0) \ln \frac{b}{a}, \quad a > 0, b > 011$$

Frullani formulasi o'rnatilgan bo'ladi.

(1) formulani ko'rsatish uchun $\lim_{x\to\infty} F(x) = B$ cheklı limitga ega bo'lgan

$$F(x) = \int_A^x \frac{f(t)}{t} dt, \quad A \leq x < \infty$$

integralini qaraymiz. U holda

$$\int_A^\infty \frac{f(ax)}{x} dx = \int_{Aa}^\infty \frac{f(t)}{t} dt = B - F(Aa),$$

$$\int_A^\infty \frac{f(bx)}{x} dx = B - F(Ab)$$

larni hosil qilamiz. O'rta qiymat haqiqiy birinchisi teoremasiga ko'ra,

$$\int_{Aa}^{Ab} \frac{f(x)}{x} dx = f(\xi) \ln x\Big|_{Aa}^{Ab} = f(\xi) \ln \frac{b}{a}, \quad \xi = Aa+\theta A(b-a), \quad 0 < \theta < 1$$

ifodaga ega bo'lamiz. $f(x)$ uzluksiz bo'lgani uchun $\lim_{x\to\infty} f(x) = f$ bo'lib,

$$\lim_{A\to\infty} \int_{Aa}^{Ab} \frac{f(ax) - f(bx)}{x} dx = \int_0^\infty \frac{f(ax) - f(bx)}{x} dx = f(0) \ln \frac{b}{a}$$

Agar $\lim_{x\to\infty} f(x) = f(\infty), f(\infty) \in R$ mavjud bo'lsa, u holda

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = (f(0) - f(\infty)) \ln \frac{b}{a} 12$$

formula o'rnatilgan bo'ladi.

Frullani formulasi tatbiqi sifatida

$$\phi(\alpha, \beta) = \int_0^\infty \frac{\sin \alpha x^4 - \sin \beta x^4}{x} dx, \quad \alpha, \beta \neq 0$$

integralni hisoblaymiz. Buning uchun $\sin \alpha x^4 - \sin \beta x^4 = \frac{1}{4}\left((1 - \cos 2\alpha x)^2 - (1 - \cos 2\beta x)^2\right) = \frac{1}{4}\left(f(|\beta|x) - f(|\alpha|x)\right)$, deb yozib, $f(x) = 2\cos 2x - \frac{1}{2}\cos 4x$ ni e'tiborga olib,

$$\phi(\alpha, \beta) = \frac{1}{4} \int_0^\infty \frac{f(|\beta|x) - f(|\alpha|x)}{x} dx = \frac{1}{4} f(0) \ln\left|\frac{\alpha}{\beta}\right| = \frac{3}{8} \ln\left|\frac{\alpha}{\beta}\right|$$

natijani hosil qilamiz.

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