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PAPER

BASIC CONCEPTS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS AND THEOREMS ON THE STABILITY OF THEIR SOLUTIONS

A. Akhmatov Azizbek [1,*] and Kh. Khusanov Djumma [2]

[1] Navoi State University
[2]Samhram" University

* a.a.akhmatov94@gmail.com

Abstract

This article examines functional-differential equations with delay and investigates the stability properties of their solutions. The main focus is placed on the characteristics of solutions in metric spaces of functions, the application of Lyapunov functionals for stability analysis, and the precompactness of shift families. The asymptotic behavior of solutions is studied through the concepts of limiting functions and limiting equations. Theorems on uniform asymptotic stability based on Lyapunov functionals are presented.

Key words: Functional-differential equations, stability, Lyapunov functional, limiting equation, asymptotic stability, metric, precompactness, space of continuous functions.

INTRODUCTION

Functional-differential equations with delay are widely used to model dynamic processes in controlled systems, biological models, engineering, and physical problems. Delay reflects real-world processes in which a system's response occurs with a time lag. The analysis of the stability of such systems is a key issue from both theoretical and applied perspectives. This work explores methods for analyzing the stability of Functional-differential equation solutions using Lyapunov functionals, as well as the concepts of precompactness, limiting functions, and limiting equations. It also investigates the properties of solutions within metric spaces defined over spaces of continuous functions. The following constructions and definitions are considered based on the works [1;2;3;4;5;6].

METODOLOGY

Let $\mathbb{R}^p$ be a real linear space of p-dimensional vectors x with norm $|x|$, and let $\mathbb{R}$ be the real axis. Let $h >$

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0 be a given real number. Denote by C the Banach space of continuous functions $\varphi : [-h, 0] \to \mathbb{R}^p$ with the norm $\|\varphi\| = \sup_{-h \le s \le 0} |\varphi(s)|$ and define $C_H = \{\varphi \in C : \|\varphi\| < H > 0\}$. Consider the functional-differential equation of delay type:

$$\dot{x}(t) = f(t, x_t), \quad f(t, 0) = 0 \tag{1}$$

where $- f : \mathbb{R} \times C_H \to \mathbb{R}^p$ is a continuous mapping satisfying the following assumptions. For a scalar $\alpha \in \mathbb{R}$ and a function $\varphi \in C_H$, the function $x = x(t, \alpha, \varphi)$ denotes the solution of equation (1) satisfying the initial condition $x_\alpha(\alpha, \varphi) = \varphi$, $x_t(\alpha, \varphi) = x(t + s, \alpha, \varphi)$ for $-h \le s \le 0$.

Assumption 1.

For each number r, where $0 < r < H$, there exists a constant $m"_r > 0$ such that the following inequality holds:

$$|f(t, \varphi)| \le m_r, \quad \forall \varphi \in \overline{C_r} = \{\varphi \in C : \|\varphi\| \le r < H\}. \tag{2}$$

Let $\{r_n\}$ be an increasing sequence of numbers such that $r_1 < r_2 < \cdots < r_n < \cdots$, $r_n \to H$ as n $\to \infty$ For each number r_i $(i = 1, 2, \ldots)$ define the set $K_i \subset C$ of functions $\varphi \in C$ such that for any s, s_1, $s_2 \in [-h, 0]$, the following inequalities hold:

$$|\varphi(s)| \le r_i, \quad |\varphi(s_2) - \varphi(s_1)| \le m_{r_i} |s_2 - s_1|.$$

The sets K_i $(i = 1, 2, \ldots)$ are compact. Define the union of all such sets:

$$\Gamma = \bigcup_{i=1}^{\infty} K_i.$$

Let F be the set of all continuous functions f, defined on $\mathbb{R} \times \Gamma$ and taking values in $\mathbb{R}^p$. Denote by f^τ the time-shift of the function f, defined as:

$$f^\tau(t, \varphi) = f(\tau + t, \varphi).$$

Note that for $f \in F$, the shift family $F_0 = \{f^\tau : \tau \in \mathbb{R}\}$ is a subset of the space F .

We introduce the notion of convergence in the space F as uniform convergence on each compact subset $K' \subset \mathbb{R} \times \Gamma$: a sequence $\{f_n \in F\}$ converges to $f \in F$, for every compact $\forall K' \subset \mathbb{R} \times \Gamma$ and for every $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that the inequality $|f_n(t, \varphi) - f(t, \varphi)| < \varepsilon$ holds for all $n > N = N(\varepsilon) > 0$ and all $(t, \varphi) \in K'$

This convergence in F is metrizable. For each $n = 1, 2, \ldots$ and for any two functions $f_1, f_2 \in F$, we define the semi norm $\|\cdot\|_n$ and the corresponding pseudometric ρ_n by the following formulas:

$$\|f\|_n = \sup\left\{|f(t, \varphi)| : \forall (t, \varphi) \in K'_n\right\}$$

$$\rho_n(f_1, f_2) = \frac{\|f_2 - f_1\|_n}{1 + \|f_2 - f_1\|_n},$$

where $K'_n = [0, n] \times K_n$, $n = 1, 2, \ldots$ was defined earlier.

The distance between two functions $f_1, f_2 \in F$ is given by:

$$\rho(f_1, f_2) = \sum_{n=1}^{\infty} 2^{-n} \rho_n(f_1, f_2). \tag{3}$$

It is easy to show that the space F satisfies all the axioms of a metric space, and that F is complete with respect to the metric defined in (3). Suppose the right-hand side of system (1.1.1) also satisfies the following assumption:

Assumption 2.

For any compact set $K \subset C_H$, the function $f = f(t, \varphi)$ is uniformly continuous with respect to $(t, \varphi) \in \mathbb{R} \times K$. That is, for any $\forall K \subset C_H$ and every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, K) > 0$ such that for any $(t, \varphi) \in \mathbb{R} \times K$, $(t_1, \varphi_1), (t_2, \varphi_2) \in \mathbb{R} \times K$ satisfying $|t_2 - t_1| < \delta$, $\varphi_1, \varphi_2 \in K$ $\|\varphi_2 - \varphi_1\| < \delta$, the inequality holds

$$|f(t_2, \varphi_2) - f(t_1, \varphi_1)| < \varepsilon. \tag{4}$$

It has been shown in [1;2] that under Assumptions 1 and 2, the family of shifts $\{f^\tau : \tau \in \mathbb{R}\}$ is precompact in F.

DATA

Definition 1.

A function $f^* : \mathbb{R} \times \Gamma \to \mathbb{R}^p$ is called a *limiting function* of if there exists a sequence $t_n \to +\infty$, such that the sequence $\{f^{(n)}(t, \varphi) = f(t_n + t, \varphi)\}$ converges to $f^*(t, \varphi)$ as $n \to \infty$ in the space F. The closure of the family $\{f^\tau : \tau \in \mathbb{R}\}$ in F is called the hull $S^+(f)$. The equation

$$\dot{x}(t) = f^*(t, x_t) \tag{5}$$

is called the *limiting equation* of (1).

Under conditions (2) and (4), equation (1.1.1) is *precompact* [1–3].

Assumption 3.

For any compact set $K \subset C_H$, the function $f = f(t, \varphi)$ satisfies a Lipschitz condition, i.e., there exists a constant $L = L(K)$ such that for all $t \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in K$ the following inequality holds:

$$|f(t, \varphi_2) - f(t, \varphi_1)| \leq L\|\varphi_2 - \varphi_1\| \tag{6}$$

If condition (6) is satisfied, then every limiting function $f^*(t, \varphi)$ also satisfies the Lipschitz condition on any compact $K \subset \Gamma$. It follows that solutions of equation (1.1.1) with initial conditions $(\alpha, \varphi) \in \mathbb{R} \times C_H$, as well as solutions of equation (1.1.5) with $(\alpha, \varphi) \in \mathbb{R} \times \Gamma$, are unique [? ?]. The following property holds for the positive limit set $\omega^+(\alpha, \varphi)$ of the $x = x(t, \alpha, \varphi)$ solution of equation (1):

Theorem 1.

Let the solution $x = x(t, \alpha, \varphi)$ of equation (1.1.1) be bounded, i.e., $x = x(t, \alpha, \varphi) \leq r < H$ for all $t \geq \alpha - h$. Then the set $\omega^+(\alpha, \varphi)$ is quasi-invariant with respect to the family of limiting equations

$$\{\dot{x} = f^*(t, x_t)\},$$

namely: for every point $\psi \in \omega^+(\alpha, \varphi)$, there exists a limiting equation such that the trajectory of its solution in the space C is contained in $\omega^+(\alpha, \varphi)$, that is,

$$\{y_t(0, \psi), t \in \mathbb{R}\} \subset \omega^+(\alpha, \varphi).$$

The direct Lyapunov method in the study of the stability of functional-differential equations is based on the use of Lyapunov functionals and functions [1]. Let us consider the application of Lyapunov functionals to the problem of stability of solutions of functional-differential equations of the form (1).

Let $V : \mathbb{R}^+ \times C_H \to \mathbb{R}$ be a continuous Lyapunov functional, and let $x = x(t, \alpha, \varphi)$ be a solution of equation (1). Then the function $V(t) = V(t, x_t(\alpha, \varphi))$ is a continuous function of time $t \geq \alpha$.

This article adopts the definition of the class K of functions of Hahn type from [], as the class of continuous functions $a \in C(\mathbb{R}^+ \to \mathbb{R}^+)$, with $a(0) = 0$, $a(s)$, $s \geq 0$, strictly increasing.

Definition 2.

The functional $V = V(t, \varphi)$ is called positive definite with respect to $|\varphi(0)|$ if $V(t, 0) \equiv 0$, and there exists H_1, with $0 < H_1 < H$, and a function $a \in K$, such that for all $(t, \varphi) \in \mathbb{R}^+ \times C(H_1)$, the inequality

$$V(t, \varphi) \geq a(|\varphi(0)|)$$

holds.

Definition 3.

The functional $V = V(t, \varphi)$ is said to admit an infinitesimal upper bound with respect to $\|\varphi\|$ if there exists H_1, with $0 < H_1 < H$, and a function $a \in K$, such that for all $(t, \varphi) \in \mathbb{R}^+ \times C(H_1)$, the inequality

$$|V(t, \varphi)| \leq a(\|\varphi\|)$$

holds.

Definition 4.

The upper right-hand derivative of the functional V along the solution $x(t, \alpha, \varphi)$ of system (1) is defined as [1]:

$$\dot{V}^+(t, x_t(\alpha, \varphi)) = \lim_{\Delta t \to 0^+} \sup \tfrac{1}{\Delta t}\left[V(t + \Delta t, x_{t+\Delta t}(\alpha, \varphi)) - V(t, x_t(\alpha, \varphi))\right].$$

For a functional admitting an invariant derivative $\dot{V}(t, \varphi)$, it is convenient to use the following expression for the derivative [1]:

$$\dot{V}(t, \varphi) = \frac{\partial V}{\partial t}(t, \varphi) + \sum_{i=1}^{p} \frac{\partial V}{\partial \varphi_i}(t, \varphi) \cdot f_i(t, \varphi) + \partial V_\varphi(t, \varphi).$$

Suppose the derivative $\dot{V}(t, \varphi)$ satisfies the inequality:

$$\dot{V}(t, \varphi) \leq -W(t, \varphi) \leq 0 \tag{7}$$

where the functional $W = W(t,\varphi)$ is bounded and uniformly continuous on each set $\mathbb{R}\times K$, i.e., for each compact set $K \subset C_H$ and for any $\varepsilon > 0$, there exist constants $m = m(K)$ and $\delta = \delta(\varepsilon, K) > 0$, such that the inequalities

$$W(t,\varphi) \le m, \quad |W(t_2,\varphi_2) - W(t_1,\varphi_1)| \le \varepsilon$$

hold for all $(t,\varphi) \in \mathbb{R}\times K$, $(t_1,\varphi_1),(t_2,\varphi_2) \in \mathbb{R}\times K$ satisfying

$$|t_2 - t_1| < \delta, \quad \varphi_1,\varphi_2 \in K, \quad \|\varphi_2 - \varphi_1\| < \delta.$$

As in the case of the function f, under the above conditions, the family of shifts

$$\{W^\tau(t,\varphi) = W(t+\tau,\varphi),\ \tau \in \mathbb{R}\}$$

is precompact in a certain space of continuous functions

$$F_W = \{W : \mathbb{R}\times\Gamma \to \mathbb{R}\}$$

equipped with the compact-open topology, which is metrizable [6].

We consider the following definitions [1].

Definition 5.

A function $W^* \subset F_W$ is called a *limiting function* of W if there exists a sequence $t_n \to +\infty$ such that the sequence

$$\{W_n(t,\varphi) = W(t_n + t,\varphi)\}$$

converges to W^* in the space F_W.

Definition 6.

We say that the solution $x(t)$, defined for $\alpha - h < t < \beta$; $\alpha < \beta \le +\infty$, of system (1), satisfies the condition

$$x_t : (\alpha,\beta) \to C_H \subset \{\varphi \in C_H : W(t,\varphi) = 0\},$$

if the identity $W(t,x_t) \equiv 0$ holds for all $t \in (\alpha,\beta)$.

Definition 7.

The functions $f^* \in F$ and $W^* \in F_W$ form a limiting pair (f^*,W^*) if they are both limiting functions of f and W, respectively, with respect to the same sequence $t_n \to +\infty$.

RESULT

Theorem 2. Suppose that:

1. There exists a continuous functional $V : \mathbb{R}\times C_H \to \mathbb{R}$ that is bounded below on each compact set $K \subset C_H$, $V(t,\varphi) \ge m(K)$ for all $(t,\varphi) \in \mathbb{R}\times K$, and satisfies $\dot V(t,\varphi) \le -W(t,\varphi) \le 0$.

2. The solution $x = x(t,\alpha,\varphi)$ of equation (1.1.1) is bounded:

$$|x(t,\alpha,\varphi)| \le H_1 < H \quad \text{for all} \quad t \ge \alpha - h.$$

Then for every limiting point $\psi \in \omega^+(\alpha,\varphi)$, there exists a limiting pair (f^*,W^*) and a solution $y(t)$ of the limiting equation (5) with initial condition $y_0 = \psi$, such that $y_t \in \omega^+(\alpha,\varphi)$ and

$$y_t \in \{\, W^*(t,\varphi) = 0\,\} \quad \text{for all} \quad (t,\varphi) \in \mathbb{R}\times C_H.$$

In the context of the asymptotic stability of the zero solution of equation (1), assuming that $f(t,0) \equiv 0$, the following result plays a fundamental role.

Theorem 3. Suppose that:

1. There exists a functional $V : \mathbb{R}\times C_{H_1} \to \mathbb{R}$ $(0 < H_1 < H)$, with $V(t,0) \equiv 0$,

$$a_1(|\varphi(0)|) \le V(t,\varphi) \le a_2(\|\varphi\|),$$

and

$$\dot V^+(t,\varphi) \le -W(t,\varphi) \le 0$$

for all $(t,\varphi) \in \mathbb{R}\times C_{H_1}$.

2. For every limiting pair (f^*,W^*), the set

$$\{\varphi \in C_H : W^*(t,\varphi) = 0\}$$

contains no solutions of the limiting equation (5) other than the zero solution.

Then the zero solution $x = 0$ of system (1) is *uniformly asymptotically stable*.

CONCLUSION

The article presents a well-founded approach to studying the stability of solutions of functional-differential equations with delay, based on the analysis of solution behavior in functional spaces. By introducing the concepts of limiting equations and limiting pairs, as well as applying Lyapunov functionals, conditions for uniform asymptotic stability have been established. The results obtained can be applied in control theory and mechanics –

particularly in analyzing the stability of complex dynamical systems involving delays. Such systems frequently arise, for example, in robotics, the dynamics of multistage mechanisms, vibrational systems, and in modeling processes with feedback. Thus, this theory serves not only as a foundation for further mathematical research but also as a powerful tool for solving practical problems related to the stability analysis of real-world engineering systems.

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