International Journal of Science and Technology

International Journal of Science and Technology, 2025, 31–34

doi: 10.70728/tech.v02.i14.007
Volume 02, Issue 14
ISSN: 3030-3443
Paper

PAPER

ON COMPLEX HYPERPLANES IN $\mathbb{C}^n$ SPACE AND THEIR PROJECTIONS IN REAL SPACE

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Abstract

This article studies the projections of complex hyperplanes in the real space of $\mathbb{C}^n$. In particular, it proves a theorem stating that for any $m \leq n$ dimensional real plane taken from the space $\mathbb{R}^n \subset \mathbb{C}^n$, there exists a unique m-dimensional complex plane that contains it.

Key words: real space, complex space, real plane, complex hyperplane.

Introduction

It is known that functions possessing subharmonic properties on the sections of m-dimensional planes in an n-dimensional real space are among the important objects of modern geometry in $\mathbb{R}^n$. In this direction, significant results have been obtained in the works of R. Harvey and B. Lawson [4–9], B. Drnovsek, B. D. Forstnerich [3], as well as other mathematicians. These papers consider a class of functions that are subharmonic on p-dimensional real planes in the Euclidean space $\mathbb{R}^n$. This class of functions is called p-plurisubharmonic functions, which were later successfully applied in p-convex geometry to describe the p-convex hulls of geometric objects.

In recent years, studying such a class of functions in $\mathbb{C}^n$, in the complex space on m-dimensional complex hyperplanes, through their subharmonic properties, has become one of the leading research directions. This, in turn, raises the problem of studying the relationship between complex hyperplanes and real planes. The results obtained in this paper, namely the study of the projections of complex hyperplanes in real space, play a significant role in solving such problems.

Research Methodology

m-dimensional hyperplanes in $\mathbb{R}^n$ (see [2]). The m hyperplanes are represented by the following

Compiled on: November 3, 2025.

equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \beta_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \beta_2, \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = \beta_m. \end{cases} \quad (1)$$

Here $x_1, x_2, \ldots, x_n$ are the coordinates of a point in $\mathbb{R}^n$, $a_{ij} \in \mathbb{R}^n$ are the coefficients representing the orientation of each hyperplane, $\beta_{ij} \in \mathbb{R}^n$ are constants determining the distance of each hyperplane from the origin.

m-dimensional real and complex hyperplanes in $\mathbb{C}^n$ (see [1]). Now suppose we are given $k \leq 2n$ vectors $a^\mu \in \mathbb{C}^n$ that are linearly independent over $\mathbb{R}$; the set of points $z \in \mathbb{C}^n$ that satisfies the following system of k real equations

$$\operatorname{Re}(z, a^\mu) = \beta_\mu, \qquad \mu = 1, \ldots, k, \quad (2)$$

is known as a real plane of codimension k, or a plane of dimension $m = 2n - k$ (for $k = 2n$, this is a point). Since k represents the number of independent constraints imposed on the system, the resulting dimension m effectively reflects the degrees of freedom remaining for the points on the plane.

If the vectors a^μ are instead linearly independent over $\mathbb{C}$ (thus requiring $k \leq n$, as k cannot exceed the dimension of the complex space), then the set described by the system of complex equations

$$(z, a^\mu) = b_\mu, \qquad \mu = 1, \ldots, k, \quad (3)$$

is called a complex plane of codimension k or dimension $m = n - k$. It follows that such a plane is determined by both the complex constraints imposed by a^μ and the specific values b_μ. Since this system can be rewritten as $2k$ real linear equations $\operatorname{Re}(z, a^\mu) = \operatorname{Re} b_\mu$, $\operatorname{Re}(z, ia^\mu) = \operatorname{Im} b_\mu$ and the vectors a^μ and ia^μ $(\mu = 1, \ldots, k)$ are linearly independent over $\mathbb{R}$, then this is a plane of real codimension $2k$.

Building on the foundational concepts of $\mathbb{R}^n$ and $\mathbb{C}^n$ spaces, we now present the main result of this study. The following theorem shows the relationship between m-dimensional hyperplanes of $\mathbb{R}^n$ and $\mathbb{C}^n$, where $\mathbb{R}^n_x \subset \mathbb{C}^n = \mathbb{R}^n_x + i\mathbb{R}^n_y$ $(z = x + iy)$.

Analysis and results

Theorem. In the space $\mathbb{C}^n$, a real m-dimensional hyperplane Π in $\mathbb{R}^n_x \subset \mathbb{C}^n$ is contained within a unique m-dimensional complex hyperplane.

Proof. Let Π be a real m-dimensional plane $(\dim \Pi = m)$ in $\mathbb{R}^n_x$. The given m-dimensional real plane is defined by the following system of equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \beta_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \beta_2, \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = \beta_m. \end{cases} \quad (4)$$

Here a_{jk} and β_j are real numbers $(j = 1, \ldots, n - m)$.

We are given the complex space which can be written as $\mathbb{C}^n = \mathbb{R}^n_x + i\mathbb{R}^n_y$, where an m-dimensional complex plane $\dim \Pi^c = m$ is given. Let this complex plane be defined by the following system of equations:

$$\begin{cases} c_{11}\bar{z}_1 + c_{12}\bar{z}_2 + \cdots + c_{1n}\bar{z}_n = \gamma_1, \\ c_{21}\bar{z}_1 + c_{22}\bar{z}_2 + \cdots + c_{2n}\bar{z}_n = \gamma_2, \\ \vdots \\ c_{m1}\bar{z}_1 + c_{m2}\bar{z}_2 + \cdots + c_{mn}\bar{z}_n = \gamma_m. \end{cases} \quad (5)$$

Here c_{jk} and γ_j are complex numbers $(j = 1, \ldots, n - m)$.

Systems (4) and (5) can be written in the following form:

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = \beta_j,$$
$$c_{j1}\bar{z}_1 + c_{j2}\bar{z}_2 + \cdots + c_{jn}\bar{z}_n = \gamma_j,$$

where $j = \overline{1, n - m}$.

$$\operatorname{Re}(c_{j1}\bar{z}_1 + c_{j2}\bar{z}_2 + \cdots + c_{jn}\bar{z}_n) = \operatorname{Re}\gamma_j$$

is equivalent to

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = \beta_j$$

according to the construction of the space $\mathbb{C}^n$.

$$\operatorname{Re}\left(\sum_{k=1}^{n} c_{jk} z_k\right) = \operatorname{Re}\left(\sum_{k=1}^{n} c_{jk}(x_k + iy_k)\right) = \sum_{k=1}^{n}(\operatorname{Re} c_{jk} x_k - \operatorname{Im} c_{jk} y_k) = \operatorname{Re}\gamma_j$$

$$\begin{cases} \sum_{k=1}^{n}(\operatorname{Re} c_{jk} x_k - \operatorname{Im} c_{jk} y_k) = \operatorname{Re}\gamma_j \\ a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n = \beta_j \end{cases}$$

to ensure that this equation is consistent, we need to equate the corresponding coefficients. For this, $\operatorname{Re} c_{jk} = a_{jk}$, $\operatorname{Im} c_{jk} = 0$ $(k = \overline{1,n},\ j = \overline{1,n-m})$ conditions must hold.

Thus, $c_{jk} = a_{jk}$ is valid.

From this, we write the following equation:

$$a_{j1}\bar{z}_1 + a_{j2}\bar{z}_2 + \cdots + a_{jn}\bar{z}_n = \gamma_j.$$

Here, $\bar{z}_n = x_n + iy_n$. Taking this into account:

$$a_{j1}(x_1 + iy_1) + a_{j2}(x_2 + iy_2) + \cdots + a_{jn}(x_n + iy_n) = \gamma_j,$$

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n + i\sum_{k=1}^{n} a_{jk}y_k = \gamma_j,$$

$$\beta_j + i\sum_{k=1}^{n} a_{jk}y_k = \gamma_j. \qquad (6)$$

Consequently, (4) aligns with (5) as: In (6), when considering the real part of the equation $i\sum_{k=1}^{n} a_{jk}y_k = 0$, it follows that $\beta_j = \gamma_j$, ensuring the consistency of the equation. This proves that the coefficients of (4) consist entirely of real numbers.

Examples of Complex Hyperplanes Containing Real Planes in $\mathbb{C}^3$. The theorem states that any real m-dimensional hyperplane in $\mathbb{R}^n_x \subset \mathbb{C}^n$ is uniquely contained within an m-dimensional complex hyperplane. In this section, we provide explicit examples following the structure of the theorem and its proof.

A real plane in $\mathbb{R}^3$ is given by the equation:

$$a_1x_1 + a_2x_2 + a_3x_3 = \beta,$$

where a_j and β are real numbers. For instance, consider:

$$x_1 + 2x_2 - x_3 = 4.$$

This equation defines a two-dimensional real hyperplane in $\mathbb{R}^3$.

We extend the equation by allowing x_j to take complex values, defining a complex hyperplane in $\mathbb{C}^3$:

$$\bar{z}_1 + 2\bar{z}_2 - \bar{z}_3 = 4$$

where $\bar{z}_j = x_j + iy_j$ are complex variables.

Expanding in terms of real and imaginary components:

$$(x_1 + iy_1) + 2(x_2 + iy_2) - (x_3 + iy_3) = 4.$$

Separating real and imaginary parts gives the system:

– Real part: $x_1 + 2x_2 - x_3 = 4,$

– Imaginary part: $y_1 + 2y_2 - y_3 = 0.$

Since the imaginary part is zero, this ensures the given real plane is uniquely contained in the complex hyperplane.

Conclusion

This paper examines a special case of the projections of complex hyperplanes in $\mathbb{C}^n$ space onto real space. The obtained result plays an important role in studying the relationships between complex hyperplanes and real planes.

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